ON THE STEINBERG PROPERTY
OF THE CONTOU-CARRÈRE SYMBOL

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ABSTRACT. The aim of this work is to show that, when A is an artinian local ring, the Contou-Carrère symbol satisfies the property of Steinberg symbols: $\langle f, 1 - f \rangle_{A((t))^\times} = 1$ for all elements $f, 1 - f \in A((t))^\times$. Moreover, we offer a cohomological characterization of the Contou-Carrère symbol from the commutator of a central extension of groups.

1. INTRODUCTION

In 1971, J. Milnor [6] defined the tame symbol d_v associated with a discrete valuation v on a field F. Explicitly, if A_v is the valuation ring, p_v is the unique maximal ideal, and $k_v = A_v/p_v$ is the residue class field, Milnor defined $d_v \colon F^\times \times F^\times \to k_v^\times$ by

$$d_v(x, y) = (-1)^{v(x) \cdot v(y)} \frac{x^{v(y)}}{y^{v(x)}} (\operatorname{mod} \, p_v).$$

(Here and below $R^\times$ denotes the multiplicative group of a ring R with unit).

J. Milnor proved that the tame symbol is a Steinberg symbol; that is, it is bimultiplicative and it satisfies the condition that $d_v(x, 1 - x) = 1$ for all $x \neq 1$.

In 1994 C. Contou-Carrère [3] defined a natural transformation greatly generalizing the tame symbol. In the case of an artinian local base ring A with maximal ideal m, the natural transformation takes the following form. Let $f, g \in A((t))^\times$ be given, where t is a variable. It is possible in exactly one way to write:

$$f = a_0 \cdot t^{w(f)} \cdot \prod_{i=1}^\infty (1 - a_i t^i) \cdot \prod_{i=1}^\infty (1 - a_{-i} t^{-i})$$
$$g = b_0 \cdot t^{w(g)} \cdot \prod_{i=1}^\infty (1 - b_i t^i) \cdot \prod_{i=1}^\infty (1 - b_{-i} t^{-i}),$$

with $w(f), w(g) \in \mathbb{Z}$, $a_i, b_i \in A$ for $i > 0$, $a_0, b_0 \in A^\times$, $a_{-i}, b_{-i} \in m$ for $i > 0$, and $a_{-i} = b_{-i} = 0$ for $i \gg 0$. By definition, the value of the Contou-Carrère

symbol is:

$$\langle f,g\rangle_{A((t))^\times} := (-1)^{w(f)w(g)} \frac{a_0^{w(g)}\prod_{i=1}^\infty\prod_{j=1}^\infty\big(1-a_i^{j/(i,j)}b_{-j}^{i/(i,j)}\big)^{(i,j)}}{b_0^{w(f)}\prod_{i=1}^\infty\prod_{j=1}^\infty\big(1-a_{-i}^{j/(i,j)}b_j^{i/(i,j)}\big)^{(i,j)}} \in A^\times.$$

The definition makes sense because only finitely many of the terms appearing in the infinite products differ from 1. The symbol $\langle\cdot,\cdot\rangle_{A((t))^\times}$ is clearly antisymmetric and, although is not immediately obvious from the definition, it is also bimultiplicative.

G. W. Anderson and the author [1] have interpreted the Contou-Carrère symbol $\langle f,g\rangle_{A((t))^\times}$—up to signs— as a commutator of liftings of f and g to a certain central extension of a group containing $A((t))^\times$, and they have exploited the commutator interpretation to prove in the style of Tate [12] a reciprocity law for the Contou-Carrère symbol on a non-singular complete curve defined over an algebraically closed field k, A being an artinian local k-algebra.

Moreover, the author has obtained a similar result for an algebraic curve over a perfect field [9], and A. Beilinson, S. Bloch and H. Esnault [2] have defined the Contou-Carrère symbol as the commutator pairing in a Heisenberg super extension. This symbol has also played an important role in a recent work by M. Kapranov and E. Vasserot [5]. In fact, since the Contou-Carrère symbol contains the classical residue, the tame symbol and the Witt residue as special cases, it is currently an important tool for studying several topics in Arithmetic Algebraic Geometry.

An open problem regarding this symbol has been to determine whether it satisfies the Steinberg property:

(1.1) $$\langle f,1-f\rangle_{A((t))^\times} = 1 \text{ for all } f,1-f\in A((t))^\times.$$

Here we solve this problem and we show that the Contou-Carrère symbol satisfies property (1.1) - Theorem 3.3 - as an application of an adjunction formula recently offered in [8], and, similar to other works by the author, we offer a cohomological characterization of this symbol from the commutator of a central extension of groups - Proposition 4.4- in the same case.

2. Preliminaries

2.1. **Witt Parameters and the Lifting Lemma.** In this work *rings* are commutative with unit. Let A be a ring. Let $A((t))$ be the ring of series of the form $\sum_{i\in\mathbb{Z}} a_i t^i$ with coefficients $a_i\in A$ such that $a_i = 0$ for $i\ll 0$. Let $A[[t]]\subset A((t))$ be the subring consisting of series of the form $\sum_{i=0}^\infty a_i t^i$. Let $A[t^{\pm 1}]\subset A((t))$ be the subring consisting of polynomials in $t^{\pm 1}$ with coefficients in A. Given an ideal $I\subset A$, let $t^{\pm 1}I[t^{\pm 1}]$ be the ideal of $A[t^{\pm 1}]$ generated by all products of the form $xt^{\pm 1}$, where $x\in I$; let $I((t))$ be the ideal of $A((t))$ consisting of series with all coefficients in I, and let $I[[t]] = A[[t]]\cap I((t))$.

Let A again be a ring. Let I be a nilpotent ideal. Let $\Gamma(A, I)$ be the set of power series $f = \sum_{i=-\infty}^{\infty} a_i t^i \in A((t))$ such that for some integer $w = w(f) = w_{A,I}(f)$ we have $a_w \in A^\times$ and $a_i \in I$ for $i < w$. The set $\Gamma(A, I)$ is closed under power series multiplication and forms a group. Let $\Gamma_0(A, I) \subset \Gamma(A, I)$ be the subgroup consisting of power series for which $w(f) = 0$. Let $\Gamma_-(A, I) \subset \Gamma_0(A, I)$ be the subgroup consisting of power series of the form $1 + f$ with $f \in t^{-1}I[t^{-1}]$. Let $\Gamma_+(A, I) = A[[t]]^\times \subset \Gamma_0(A, I)$. Given pairs (A, I) and (B, J), each consisting of a ring and a nilpotent ideal, and a ring homomorphism $\varphi : A \to B$ such that $\varphi(I) \subset J$, the corresponding group homomorphism $\Gamma(\varphi) : \Gamma(A, I) \to \Gamma(B, J)$ is defined to be that sending $\sum_i a_i t^i$ to $\sum_i \varphi(a_i) t^i$. Thus, the construction Γ becomes a functor, and similarly the related constructions $\Gamma_\pm$ and Γ_0 become functors.

For a detailed study of the functor Γ, the reader is referred to [7].

Lemma 2.1. *Let A be a ring. Let $I \subset A$ be a nilpotent ideal. Let $f \in \Gamma_0(A, I)$ be given. Then there exist unique $g \in \Gamma_+(A, I)$ and $h \in \Gamma_-(A, I)$ such that $f = gh$.*

Proof. Let ν be a positive integer such that $I^\nu = 0$. We proceed by induction on ν. For $\nu = 1$ there is nothing to prove, so we assume that $\nu > 1$ for the rest of the proof. Write $f = f_+ + f_-$, where $f_+ \in A[[t]]^\times$ and $f_- \in t^{-1}I[t^{-1}]$, in the unique possible way. After replacing f by $(f_+)^{-1}f$, we may assume without loss of generality that $f = 1 - \widetilde{f}$, where $\widetilde{f} \in I((t))$. Write $\widetilde{f} = \widetilde{f}_+ + \widetilde{f}_-$, where $\widetilde{f}_+ \in I[[t]]$ and $\widetilde{f}_- \in t^{-1}I[t^{-1}]$, in the only possible way. After replacing f by

$$\left(1 + \sum_{i=1}^{\nu-1} (\widetilde{f}_-)^i\right)(1 - \widetilde{f}_+)^{-1}f = (1 - \widetilde{f}_-)^{-1}(1 - \widetilde{f}_+)^{-1}f,$$

we may assume without loss of generality that $1 - f \in I^2((t))$, in which case we are finished by induction on ν. $\qquad\square$

Lemma 2.2. *Let A be a ring. Let $f \in A[[t]]^\times$ be given. Then:*
(i) There exists a unique sequence $\{a_i\}_{i=1}^\infty$ in A such that $f = f(0)\prod_{i=1}^\infty(1 - a_i t^i)$.
(ii) If $f = 1 + t^{n+1}g$ for some $g \in A[[t]]$ and positive integer n, then $a_1 = \cdots = a_n = 0$.
(iii) If there exists a nilpotent ideal $I \subset A$ such that $1 - f \in tI[t]$, then $a_i \in I$ for all i and $a_i = 0$ for all $i \gg 0$.

Proof. (i) Write $f/f(0) = 1 - tg_1$ for some $g_1 \in A[[t]]$. For $i > 0$ there exists unique $g_i \in A[[t]]$ such that

$$1 - t^i g_i = (1 - g_{i-1}(0)t^{i-1})^{-1}(1 - t^{i-1}g_{i-1}).$$

Put $a_i = g_i(0)$ for all i. The resulting sequence $\{a_i\}_{i=1}^\infty$ is the only possible one with the desired properties.

(ii) The method of construction of the sequence $\{a_i\}_{i=1}^\infty$ also proves this.

(iii) Let ν be a positive integer such that $I^\nu = 0$. If $\nu = 1$, there is nothing to prove, so we assume that $\nu > 1$ for the rest of the proof. Write $f = 1 + \sum_{i=1}^{\mu} b_i t^i$ for some coefficients $b_i \in I$ and positive integer μ. By the uniqueness asserted in part (i) we have congruences $a_i \equiv b_i \mod I^2$ for $i = 1, \dots, \mu$ and $a_i \equiv 0 \mod I^2$ for $i > \mu$. Consider $f^* = f \prod_{i=1}^{\mu} (1 + \sum_{j=1}^{\nu-1} a_i^\nu t^{i\nu}) = f \prod_{i=1}^{\mu} (1 - a_i t^i)^{-1}$. Then, $f^* = 1 - \sum_{\mu < i \le \mu^*} b_i^* t^i$ for some integer $\mu^* \ge \mu$ and coefficients $b_i^* \in I^2$. Writing $f^* = \prod_{i=1}^{\infty} (1 - a_i^* t^i)$ according to part (i) for unique coefficients $a_i^* \in A$, we have $a_i^* = 0$ for $i \gg 0$ by induction on ν. We further have $a_1^* = \cdots = a_\mu^* = 0$ by part (ii), and finally we have $a_i^* = a_i$ for $i > \mu$ by the uniqueness asserted in part (i). $\quad\square$

Proposition 2.3. *Let A be a ring. Let $I \subset A$ be a nilpotent ideal. Let $f \in \Gamma(A, I)$ be given. Then there exist unique coefficients $\{a_i\}_{i=-\infty}^{\infty}$ in A such that $a_0 \in A^\times$, $a_i \in I$ for $i < 0$, $a_i = 0$ for $i \ll 0$, and $f = t^{w(f)} a_0 (\prod_{i=1}^{\infty} (1 - a_i t^i))(\prod_{i=1}^{\infty} (1 - a_{-i} t^{-i}))$.*

We call $\{a_i\}_{i=-\infty}^{\infty}$ the family of *Witt parameters* $f \in \Gamma(A, I)$.

Proof. We combine the preceding two lemmas. $\quad\square$

We should note that the Witt parameters $\{a_i\}_{i=-\infty}^{\infty}$ depend functorially on f, i. e., given a ring homomorphism $\varphi : A \to A'$ and nilpotent ideals $I \subset A$ and $I' \subset A'$ such that $\varphi(I) \subset I'$, if $\{a_i\}_{i=-\infty}^{\infty}$ are the Witt parameters in A of $f \in \Gamma(A, I)$, then $\{\varphi(a_i)\}_{i=-\infty}^{\infty}$ are the Witt parameters in A' of $\Gamma(\varphi)(f) \in \Gamma(A', I')$.

Lemma 2.4 (Lifting Lemma)**.** *Let A be a ring and let $I \subset A$ be a nilpotent ideal. Let $f \in A((t))$ be given such that $f, 1 - f \in \Gamma_0(A, I)$. We can then find: a ring A_1 and nilpotent ideal $I_1 \subset A_1$; an artinian local $\mathbb{Q}$-algebra A_2 and nilpotent ideal $I_2 \subset A_2$; a ring homomorphism $\varphi : A_1 \to A$ such that $\varphi(I_1) \subset I$; an element $f_1 \in A_1((t))$ such that $f_1, 1 - f_1 \in \Gamma_0(A_1, I_1)$, $\Gamma(\varphi)(f_1) = f$, and $\Gamma(\varphi)(1 - f_1) = 1 - f$, and a ring homomorphism $\psi : A_1 \to A_2$ such that $\psi(I_1) \subset I_2$ and $\Gamma(\psi)$ is injective.*

Proof. Let us write $f = \sum_{i \in \mathbb{Z}} b_i t^i$ where $b_i \in A$, $b_i \in I$ for $i < 0$, $b_i = 0$ for $i \ll 0$. Let us choose a positive integer N such that $f = \sum_{i \ge -N} b_i t^i$ and $b_i^N = 0$ for $i < 0$. Let Λ_0 be the polynomial ring over $\mathbb{Z}$ generated by a family of independent variables $\{X_i\}_{i \in \mathbb{Z}}$, and let Λ be obtained from Λ_0 by inverting X_0 and $1 - X_0$. Let $J_0 \subset \Lambda$ be the ideal generated by

$$\{X_i\}_{i < -N} \cup \{X_i^N\}_{-N \le i \le -1}$$

and let $J_1 \subset \Lambda$ be the ideal generated by $\{X_i\}_{i < 0}$. Let $A_1 = \Lambda/J$ and $I_1 = J_0/J_1$. Let A_2 be the ring obtained from A_1 by inverting all non-zero-divisors (the total quotient ring of A_1) and let I_2 be the ideal of A_2 generated by fractions with numerator in I_1. Let φ be the unique homomorphism sending X_i to a_i for all i. Let $f_1 = \sum_i X_i t^i \in \Gamma(A_1, I_1)$. Let $\psi : A_1 \to A_2$ be the natural inclusion. Then the objects thus constructed have all desired properties. $\quad\square$

2.2. Commensurability of A-modules and the Contou-Carrère symbol.

Let A now be an artinian local ring with maximal ideal m. Let V_A be a free A-module. Given free A-submodules $E, F \subset V$, we write $E \sim F$ and say that E and F are *commensurable* if the quotient $\frac{E+F}{E \cap F}$ is finitely generated over A. It is easily verified that commensurability is an equivalence relation.

Given a free A-submodule $V_A^+ \subset V_A$, let $\mathrm{Gl}(V_A, V_A^+)$ denote the set of A-linear automorphisms σ of V_A such that $V_A^+ \sim \sigma V_A^+$. It is easily verified that $\mathrm{Gl}(V_A, V_A^+)$ is a subgroup of the group of A-linear automorphisms of V_A depending only on the commensurability class of V_A^+.

Using the theory of groupoids, we can construct a group $\widetilde{\mathrm{Gl}}(V_A, V_A^+)$ depending only on the commensurability class of V_A^+ and such that there exists a central extension of groups:

$$(2.1) \qquad 1 \to A^\times \to \widetilde{\mathrm{Gl}}(V_A, V_A^+) \xrightarrow{\pi} \mathrm{Gl}(V_A, V_A^+) \to 1 \,.$$

We denote by $\{\cdot, \cdot\}_{V_A}^{V_A^+}$ the commutator of the central extension (2.1); that is, if τ and σ are two commuting elements of $\mathrm{Gl}(V_A, V_A^+)$ and $\widetilde{\tau}, \widetilde{\sigma} \in \widetilde{\mathrm{Gl}}(V_A, V_A^+)$ are elements such that $\pi(\widetilde{\tau}) = \tau$ and $\pi(\widetilde{\sigma}) = \sigma$, then one has a commutator pairing:

$$\{\tau, \sigma\}_{V_A}^{V_A^+} = \widetilde{\tau} \cdot \widetilde{\sigma} \cdot \widetilde{\tau}^{-1} \cdot \widetilde{\sigma}^{-1} \in A^\times \,.$$

If we set $V_A^+ = A[[t]]$ and $V_A = A((t))$, we have that $A((t))^\times \subseteq \mathrm{Gl}(V_A, V_A^+)$ (viewing a Laurent series f as a homothety) and $A((t))^\times$ is a commutative group. From this immersion of groups, the central extension (2.1) induces a new central extension of groups:

$$(2.2) \qquad 1 \to A^\times \to \widetilde{A((t))^\times} \to A((t))^\times \to 1 \,,$$

and we have a commutator map:

$$\{\cdot, \cdot\}_{A((t))}^{A[[t]]} : A((t))^\times \times A((t))^\times \longrightarrow A^\times \,.$$

Given $f \in A((t))^\times$, by considering the family of *Witt parameters* of f (Proposition 2.3), one has a unique presentation:

$$f = a_0 \cdot t^{w(f)} \cdot \prod_{i=1}^\infty \left(1 - a_{-i} t^{-i}\right) \cdot \prod_{i=1}^\infty \left(1 - a_i t^i\right) \,,$$

where

$$w(f) \in \mathbb{Z}, \quad \begin{cases} a_i = 0 & \text{if } i \ll 0, \\ a_i \in m & \text{if } i < 0, \\ a_i \in A^\times & \text{if } i = 0, \\ a_i \in A & \text{if } i > 0. \end{cases}$$

The integer number $w(f)$ is the *winding number* of f.

Thus, if $g \in A((t))^\times$ is another element with presentation

$$g = b_0 \cdot t^{w(g)} \cdot \prod_{i=1}^\infty (1 - b_i t^i) \cdot \prod_{i=1}^\infty (1 - b_{-i} t^{-i}) \,,$$

such that the coefficients satisfy the above conditions, a computation shows that the value of the commutator is:

$$\{f,g\}_{A((t))}^{A[[t]]} = \frac{a_0^{w(g)}\prod_{i=1}^{\infty}\prod_{j=1}^{\infty}\big(1 - a_i^{j/(i,j)}b_{-j}^{i/(i,j)}\big)^{(i,j)}}{b_0^{w(f)}\prod_{i=1}^{\infty}\prod_{j=1}^{\infty}\big(1 - a_{-i}^{j/(i,j)}b_j^{i/(i,j)}\big)^{(i,j)}} \in A^{\times}\,.$$

This expression makes sense because only finitely many of the terms appearing in the infinite products differ from 1, and the Contou-Carrère symbol [3] is:

$$\langle f,g\rangle_{A((t))^{\times}} = (-1)^{w(f)w(g)} \cdot \{f,g\}_{A((t))}^{A[[t]]}\,.$$

For arbitrary elements $f,g,h \in A((t))^{\times}$, the following relations hold:

- $\langle f, g \cdot h\rangle_{A((t))^{\times}} = \langle f,g\rangle_{A((t))^{\times}} \cdot \langle f,h\rangle_{A((t))^{\times}}$.
- $\langle g,f\rangle_{A((t))^{\times}} = \langle f,g\rangle_{A((t))^{\times}}^{-1}$.
- $\langle f,-f\rangle_{A((t))^{\times}} = 1$.
- Given $\varphi \in A((t))^{\times}$ with positive winding number n, one has that:

(2.3) $$\langle f, g \circ \varphi\rangle_{A((t))^{\times}} = \langle \mathcal{N}_{\varphi}[f],g\rangle_{A((t))^{\times}}\,,$$

where $\mathcal{N}_{\varphi}\colon A((t))^{\times} \to A((t))^{\times}$ denotes the corresponding norm mapping -viewing $A((t))$ via the homomorphism $h \mapsto h \circ \varphi$ as a free $A((t))$-module of rank n- ([8], Proposition 3.6). As an application of this "adjunction formula" one has that the Contou-Carrère symbol is invariant under reparameterization of $A((t))$ in the following sense: if $\tau \in A((t))$ is a element with winding number equal to 1, then $\langle f,g\rangle_{A((t))^{\times}} = \langle f \circ \tau, g \circ \tau\rangle_{A((t))^{\times}}$.

3. The Steinberg property of the Contou-Carrère symbol

With the above notation, let us now consider a Laurent series

$$f = \sum_{i \geq -N} a_i t^i \in A((t))^{\times}\,.$$

Lemma 3.1. *If $w(f) = w(1-f) = 0$ and $a_i \in A^{\times}$ for a positive integer $i > 0$, then there exists an invertible element $\lambda \in A^{\times}$ and a series $\varphi \in A((t))^{\times}$ with $w(\varphi) > 0$ such that*

$$1 - f = (1-\lambda)(1-\varphi)\,.$$

Proof. With the conditions of the Lemma, it is clear that there exists an invertible element $\lambda \in A^{\times}$ and a series $g \in A((t))^{\times}$ with $w(g) > 0$ such that $f = \lambda \cdot (1 + g)$.

Thus,

$$1 - f = (1-\lambda)[1 - \frac{\lambda g}{1-\lambda}]\,,$$

and writing $\varphi = \frac{\lambda g}{1-\lambda}$ the claim is deduced. $\square$

With the notation of the preceding Lemma, note that $f = \lambda + (1-\lambda)\varphi$.

Lemma 3.2. *If A is an artinian local $\mathbb{Q}$-algebra, and $\widetilde{f} \in A((t))$ is a nilpotent element, one has that*

$$\langle 1 + \widetilde{f}, 1 + \mu\widetilde{f} \rangle_{A((t))^\times} = 1$$

for all $\mu \in A^\times$.

Proof. Recall from ([4], p. 154) that if A is a $\mathbb{Q}$-algebra and $g \in 1 + m((t))$, then

$$\langle f, g \rangle_{A((t))^\times} = \exp(\operatorname{Res}_{t=0}[\log g \cdot d\log f])$$

for all $g \in A((t))^\times$.

Thus, the statement of the Lemma follows from the well-known property of residues:

$$\operatorname{Res}_{t=0}[\widetilde{f}^n d\widetilde{f}] = 0\,,$$

for every $\widetilde{f} \in A((t))$ and $n \geq 0$. $\qquad\square$

Theorem 3.3. *[Steinberg property] If A is an artinian local ring, given an element $f \in A((t))^\times$ such that $1 - f \in A((t))^\times$, one has that:*

$$\langle f, 1 - f \rangle_{A((t))^\times} = 1\,.$$

Proof. As in the proof of J. Milnor [6] related to the tame symbol, the proof of this theorem will be divided into several cases.

If $w(f) = n > 0$, from expression (2.3) one has that:

$$
\begin{aligned}
\langle 1 - f, f \rangle_{A((t))^\times} &= \langle 1 - f, t \circ f \rangle_{A((t))^\times} \\
&= \langle \mathcal{N}_f[1 - f], t \rangle_{A((t))^\times} \\
&= \langle (1 - t)^n, t \rangle_{A((t))^\times} = 1\,,
\end{aligned}
$$

and the claim is deduced in this case.

Moreover, when $w(f) < 0$, bearing in mind that $\langle f^{-1}, -f^{-1} \rangle_{A((t))^\times} = 1$, from the above result we have that:

$$
\begin{aligned}
\langle f, 1 - f \rangle_{A((t))^\times} &= \langle f^{-1}, 1 - f \rangle_{A((t))^\times}^{-1} \cdot \langle f^{-1}, -f^{-1} \rangle_{A((t))^\times}^{-1} \\
&= \langle f^{-1}, (1 - f)(-f^{-1}) \rangle_{A((t))^\times}^{-1} \\
&= \langle f^{-1}, 1 - f^{-1} \rangle_{A((t))^\times}^{-1} = 1\,.
\end{aligned}
$$

Furthermore, if $w(f) = w(1 - f) = 0$, and f satisfies the condition of Lemma 3.1, with the notation of this Lemma one has that:

$$
\begin{aligned}
\langle f, 1 - f \rangle_{A((t))^\times} &= \langle f, 1 - \lambda \rangle_{A((t))^\times} \cdot \langle f, 1 - \varphi \rangle_{A((t))^\times} \\
&= \langle f, (1 - t) \circ \varphi \rangle_{A((t))^\times} = \langle \mathcal{N}_\varphi[f], 1 - t \rangle_{A((t))^\times} \\
&= \langle [\lambda + (1 - \lambda)t]^{w(\varphi)}, 1 - t \rangle_{A((t))^\times} = 1\,.
\end{aligned}
$$

Finally, if $w(f) = w(1 - f) = 0$, and $a_i \in m$ for all $i \neq 0$, it follows from the Lifting Lemma (Lemma 2.4) that we can assume without loss of generality that A is an artinian local $\mathbb{Q}$-algebra. Then, writing $f = a_0(1 + \widetilde{f})$ we have that $1 - f = (1 - a_0)(1 - \frac{a_0}{1 - a_0}\widetilde{f})$.

Hence, setting $\mu = -\frac{a_0}{1-a_0}$, we conclude from Lemma 3.2 bearing in mind that:

$$\langle f, 1-f \rangle_{A((t))^\times} = \langle 1+\widetilde{f}, 1+\mu\widetilde{f} \rangle_{A((t))^\times} = 1 \,.$$

$\square$

4. Cohomological characterization of the Contou-Carrère symbol

Let $\{\cdot,\cdot\}_{A((t))}^{A[[t]]}$ again be the commutator of the central extension of groups (2.2). Since $\{\cdot,\cdot\}_{A((t))}^{A[[t]]}$ is a 2-cocycle, it determines an element of the cohomology group $H^2(A((t))^\times, A^\times)$ -[11], page 168-.

Similar to above, we can say that a map $\psi\colon A((t))^\times \times A((t))^\times \longrightarrow A^\times$ is called a "Steinberg map" when:

- ψ is bimultiplicative.
- $\psi(f, 1-f) = 1$ for all $f \in A((t))^\times$ such that $1-f \in A((t))^\times$.

Remark 4.1. A Steinberg map ψ also satisfies the following properties:

- $\psi(f,g) = [\psi(g,f)]^{-1}$;
- $\psi(f,-f) = 1$,

for all $f, g \in A((t))^\times$.

Remark 4.2. The commutator $\{\cdot,\cdot\}_{A((t))}^{A[[t]]}$ is not a Steinberg map because

$$\{\frac{1}{t}, 1 - \frac{1}{t}\}_{A((t))}^{A[[t]]} = -1 \,.$$

We shall now give a cohomological characterization of the Contou-Carrère symbol from the cohomology class $[\{\cdot,\cdot\}_{A((t))}^{A[[t]]}] \in H^2(A((t))^\times, A^\times)$.

Recall that given two commutative groups, G and B, $H^2(G,B)$ is the group of classes of 2-cocycles $c\colon G \times G \to B$ (mod. 2-coboundaries), where a 2-cocycle b is a 2-coboundary when there exists a map $\phi\colon G \to B$ such that

$$b(\alpha,\beta) = (\delta\phi)(\alpha,\beta) = \phi(\alpha \cdot \beta) \cdot \phi(\alpha)^{-1} \cdot \phi(\beta)^{-1} \,.$$

Similar to [10], one has that:

Lemma 4.3. *There exists a unique 2-coboundary*

$$b\colon \mathbb{Z} \times \mathbb{Z} \to A^\times$$

satisfying the conditions:

- $b(\alpha, \beta+\gamma) = b(\alpha,\beta) \cdot b(\alpha,\gamma)$;
- $b(\alpha,\alpha) = (-1)^\alpha$,

for all $\alpha, \beta, \gamma \in \mathbb{Z}$.

Proof. Let $\phi(\alpha) = \lambda_\alpha \in A^\times$. If $b = \delta\phi$, it follows from the above conditions that

$$\lambda_\alpha = (-1)^{\frac{\alpha(\alpha-1)}{2}} \lambda_1^\alpha \qquad \text{for each } \alpha \in \mathbb{Z} \,,$$

and $b(\alpha, \beta) = (-1)^{\alpha \cdot \beta}$ is therefore the unique 2-coboundary that satisfies the required conditions. $\square$

Proposition 4.4 (Cohomological characterization of the Contou-Carrère symbol)**.** *If A is an artinian local ring, there exists a unique Steinberg map $\langle \quad , \quad \rangle_{A((t))^\times}$ in the cohomology class*

$$[\{\cdot, \cdot\}^{A[[t]]}_{A((t))}] \in H^2(A((t))^\times, A^\times)$$

satisfying the condition:

$$\langle f, g \rangle_{A((t))^\times} = \{f, g\}^{A[[t]]}_{A((t))} \ \ if \ w(f) = 0$$

for all $f, g \in A((t))^\times$.

This element is the Contou-Carrère symbol $\langle \quad , \quad \rangle_{A((t))^\times}$.

Proof. Let $\nu(f, g) = c'(f, g) \cdot \{f, g\}^{A[[t]]}_{A((t))}$ be an element of the cohomology class $[\{\cdot, \cdot\}^{A[[t]]}_{A((t))}] \in H^2(A((t))^\times, A^\times)$ satisfying the condition of the proposition. Since c' is a 2-coboundary, one has that $c'(f, g) = 1$ when $w(g) = 0$ and, therefore, there exists a commutative diagram

$$
\begin{array}{ccc}
A((t))^\times \times A((t))^\times & & \\
\Big\downarrow{\scriptstyle w \times w} & \searrow{\scriptstyle c'} & \\
\mathbb{Z} \times \mathbb{Z} & \xrightarrow{\ \widetilde{c}'\ } & A^\times
\end{array}
$$

where $\widetilde{c}'$ is a 2-coboundary satisfying

$$\widetilde{c}'(x, y + z) = \widetilde{c}'(x, y) \cdot \widetilde{c}'(x, z).$$

Furthermore, since ν is a Steinberg map, then $\nu(f, -f) = 1$, and one has that $\widetilde{c}'(x, x) = (-1)^x$.

It then follows from Lemma 4.3 that $\widetilde{c}'(x, y) = (-1)^{x \cdot y}$ and

$$c'(f, g) = (-1)^{w(f) \cdot w(g)}.$$

Thus, the unique element in $[\{\cdot, \cdot\}^{A[[t]]}_{A((t))}] \in H^2(A((t))^\times, A^\times)$ satisfying the condition of the proposition is $\nu(f, g) = (-1)^{w(f) \cdot w(g)} \cdot \{f, g\}^{A[[t]]}_{A((t))}$ and $\nu(f, g) = \langle f, g \rangle_{A((t))^\times}$. $\square$

Remark 4.5. The above property, which characterizes the Contou-Carrère symbol, is equivalent to one of the conditions that J.P. Serre ([11]) gave to define local symbols on algebraic curves that are also Steinberg symbols.

In the formalism of the cohomological characterization of other classical symbols, the author has formulated a conjecture to replace this condition with the continuity of the map by considering natural topologies on the corresponding groups G and B that determine the cohomology group $H^2(G, B)$. However, in this case we are not sufficiently confident to formulate a conjecture about a finer characterization of the Contou-Carrère symbol.

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